Art - 3 Definition of $P_n(x)$ and $Q_n(x)$

The solution of Legendre's equation is called Legendre's function.

When n is a positive Integer and $a_0 = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}$

the solution ⑤ of Art ② is denoted by $P_n(x)$

is called Legendre's function of Ist kind.

$$\therefore P_n(x) = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-2)}x^{n-4} - \cdots\right]$$

is called Legendre's Polynomials for different values of n, Since $P_n(x)$ is a terminating series.

We can write

$$P_n = \sum_{r=0}^{n/2} (-1)^r \frac{(2n-2r)!}{2^n r! (n-2r)! (n-r)!} x^{n-2r}$$

where $\left(\frac{n}{2}\right) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$

Again when n is positive Integer and

$\left[a_0 = \frac{n!}{1\cdot3\cdot5 \cdots (2n+1)} \right.$ the solution ⑥

of Art 2 is denoted by $Q_n(x)$ and is called Legendre's function of 2nd kind.

$$Q_n(x) = \frac{n!}{1\cdot3\cdot5 \cdots (2n+1)} \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4(2n+3)(2n+5)} x^{-n-5} + \cdots \right.$$

$Q_n(x)$ is an Infinite or non-terminating Series as n is positive

Art-4 General solution.

The most general solution of the legendre's equation is

$y = A P_n(x) + B Q_n(x)$ where A and B

are arbitrary constants.

Art-5 To show that $P_n(x)$ is the coeff of h^n in the expansion in ascending powers of $(1-2hx+h^2)^{-1/2}$

→ we have

$$(1-2xh+h^2)^{-1/2} = \{1-h(2x-h)\}^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1\cdot3}{2\cdot4}h^2(2x-h)^2 + \cdots$$

$$+ \frac{1\cdot3\cdots(2n-3)}{2\cdot4\cdots(2n-2)}h^{n-1}(2x-h)^{n-1}$$

$$+ \frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}h^n(2x-h)^n + \cdots$$

coeff of h^n

$$= \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n}(2x)^n + \frac{1\cdot 3 \cdots (2n-3)}{2\cdot 4 \cdots (2n-2)}\, {}^{n-1}C_1 (2x)^{n-2}$$

$$= \frac{1\cdot 3 \cdots (2n-1)}{n!}\left[x^n - \frac{2n}{2n-1}\cdot(n-1)\frac{x^{n-2}}{2^2} + \frac{2n(2n-2)}{(2n-1)(2n-3)}\cdot\frac{(n-2)(n-3)}{2!}\cdot\frac{x^{n-4}}{2^4} - \cdots\right]$$

$$= \frac{1\cdot 3 \cdots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2\cdot(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-2)}\cdot x^{n-4} - \cdots\right]$$

$$= P_n(x)$$

we can say that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$$